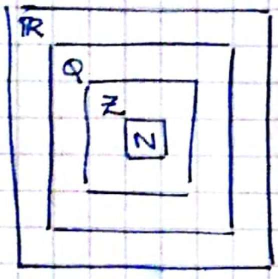
04/02/18

$\mathbb{N}$ – set of all natural integers (including 0)
$\mathbb{Z}$ – set of all integers
$\mathbb{Q}$ – set of all rational numbers ($\frac{a}{b}$, $a,b \in \mathbb{Z}$) ($b \neq 0$)

$\mathbb{R}$ – set of all real numbers (includes rational + irrational)



Set A, $A^* = A \setminus \{0\}$
eg $\mathbb{N}^* = \{1, 2, 3, 4 \ldots\}$
$\mathbb{Z}^* = \{\ldots -3, -2, -1, 1, 2 \ldots\}$
$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Def: $\boxed{1}$ Let $n \in \mathbb{Z}$
We say n is an even integer if $n = 2k$ for some $k \in \mathbb{Z}$

$\boxed{2}$ We say n is odd if $n = 2k+1$ for some $k \in \mathbb{Z}$

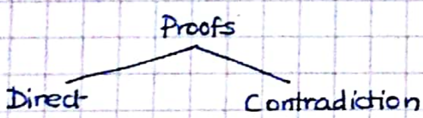
eg: $-15 = 2(-8) + 1$, $k = -8$ $\qquad 30 = 2(15)$, $k = 15$

Fact: every ~~fraction~~ rational number $\frac{a}{b}$ can be written in reduced form.

s.t. $\frac{a}{b} = \frac{c}{d}$ where $\gcd(c,d) = 1$.

eg $\frac{4}{10} = \frac{2}{5}$, $\gcd(2,5) = 1$. Because every $ integer can be written as product of a prime and cancel common factors. i.e $\frac{4}{10} = \frac{2 \times 2}{2 \times 5} = \frac{2}{5}$

PROVING.



Q: Prove ~~Proof~~ that the sum of any 2 even integers ~~is~~ is an even integer.
Proof: (Direct) given: 2 even integers. show: sum is even integer.
let x, y be even integers.
Hence by definition, $x = 2k_1$, $y = 2k_2$ for some $k_1, k_2 \in \mathbb{Z}$.
We show $x+y$ is even. (show $x+y = 2m$ for some $m \in \mathbb{Z}$)

$x+y = 2k_1 + 2k_2 = 2(k_1+k_2)$, $k_1+k_2 = m \in \mathbb{Z}$ (integer + integer = integer)

$\therefore$ $x+y = 2m$, $m \in \mathbb{Z}$.
Thus, $x+y$ is an even integer.

Q: Prove that $x = 1.322\overline{2}\ldots$ is a rational number.
Proof: (idea, do some basic algebra so that only the repeated digit is to the right of the decimal point).

$$x = 1.3\overline{2} = 1.3222\ldots$$
$$10x = 13.222\ldots$$
$$10x - x = \begin{array}{r} 13.222\ldots \\ -\ 1.322\ldots \\ \hline 11.900\ldots \end{array}$$
$$9x = 11.9$$
$$x = \frac{11.9}{9} = \frac{119}{90}, \quad x \in \mathbb{Q}, \quad x = \frac{a}{b}, \quad a,b \in \mathbb{Z}.$$

Q: Prove $x = \overset{2.015}{\cancel{2.11313\overline{13}}}$ is a rational number.
Proof: ~~$x = 2.11313\overline{13}$~~
~~$= 2.1131313\ldots$~~

$$x = 2.0151313\overline{13}$$
$$= 2.015131313\ldots$$
$$1000x = 2015.131313\ldots$$
$$1000x - x = \begin{array}{r} 2015.13131313\ldots \\ -\ 2.015131 \end{array}$$

man.

Q: Prove $x = 2.01\overline{13}$ is rational.
Proof: $x = 2.01131313\ldots$
$$100x = 201.131313\ldots$$
$$100x - x = \begin{array}{r} 201.13131313\ldots \\ -\ 2.01131313\ldots \\ \hline \end{array}$$
$$99x = 199.12$$

$$x = \cancel{149}\ \frac{199.12}{99} = \frac{19912}{9900} = \frac{4978}{2475} = \frac{a}{b}, \quad a,b \in \mathbb{Z}.$$

06/02/18 Q: Prove $1.3\overline{25}$ is a rational number.
Proof. $x = 1.3252525\ldots$
$$100x = 132.5252525\ldots$$

$$100x - x = \begin{array}{r} 132.52525\,25\ldots \\ -\ 1.32525\,25\ldots \\ \hline 131.2 \end{array}$$

$$x = \frac{131.2}{99}$$
$$x = \frac{656}{495} = \frac{a}{b}, \cancel{\emptyset} \quad a,b \in \mathbb{Z}$$

Q: Prove $7.251\overline{265}$ is rational.
Proof: $x = 7.251265265265\ldots$

$$1000x = 7251.265265265\ldots$$
$$1000x - x = \begin{array}{r} 7251.265265265\ldots \\ -\ 7.251265265\ldots \\ \hline 7244.014 \end{array}$$
$$x = \frac{7244.014}{999} = \frac{7244014}{999000}$$

Q : $1.00\overline{421}$

Proof: $x = 1.00421421421\ldots$

$1000x = 1004.21421421\ldots$

$1000x - x = 1004.21421421421\ldots$

$- \; 1.00421421421\ldots$

1003.21

$x = \dfrac{1003.21}{999} = \dfrac{100321}{99900}$

Q : $7.21265265265\ldots$ do at home.

Q : $1.53871\overline{23}$.

Proof : $x = 1.5387123\,23\,23\ldots$

$100x = 153.87123\,23\,23\ldots$

$100x - x = 1.53871'23\,23\,23\ldots$

$- \; 1.5387123\;23\ldots$

152.33252

$x = \dfrac{152.33252}{99}$

$x = \dfrac{15233252}{9900000} = \dfrac{a}{b}, \; a, b \in \mathbb{Z}$

Fix an integer $m \in \mathbb{Z}$.

Q : Convince me that $\exists \; c_1, c_2 \in \mathbb{Z}$ s.t

$231792 = 19c_1 + 24c_2$

$m = 19c_1 + 24c_2$

Proof: Since $\gcd(19, 24) = 1$, $\exists \; b_1, b_2 \in \mathbb{Z}$ s.t $1 = 19b_1 + 24b_1$

Now, multiply by m: $m = 19b_1 m + 24b_2 m$.

where $b_1 m = c_1$ and $b_2 m = c_2$

Hence, $m = 19c_1 + 24c_2$, $c_1, c_2 \in \mathbb{Z}$.

This is direct proof.

Q: $a, b \in \mathbb{N}^*$ (without 0)

$20 = ac_1 + bc_2$ for some $c_1, c_2 \in \mathbb{Z}$

i) can we conclude something on gcd between a and b?
if so, prove your claim.

Proof : Yes, $\gcd(a,b)$ is a factor of 20. i.e $\gcd(a,b) \mid 20$.

let $m = \gcd(a,b)$. Then $m \mid 20$.

Since $m \mid a$ and $m \mid b$ we conclude that $m \mid ac_1 + bc_2$. Thus, m is a factor of 20. i.e $m \mid 20$.

more generally, $L = ac_1 + bc_2$, $a, b \in \mathbb{N}^*$, $c_1, c_2 \in \mathbb{Z}$

So, $\gcd(a,b) = m$.

Since $m \mid a$ and $m \mid b$, we conclude that $m \mid ac_1 + bc_2$. Thus, $m \mid L$.

Homework.

1. Let n, m be distinct prime integers. Convince me that there exists $c_1, c_2 \in \mathbb{Z}$ such that $2018 = nc_1 + mc_2$

2. Let $m, a, b \in \mathbb{Z}$. Assume $m|a$ and $m|b$. Convince me that for every integers i_1, i_2, we have $m|(ai_1 + bi_2)$. i.e show that $ai_1 + bi_2 = mk$ for some $k \in \mathbb{Z}$

3. Let $x, y \in \mathbb{Q}$. Prove that $x+y$ is rational. i.e show that $x+y = \frac{c}{d}$ for some integers $c \in \mathbb{Z}$ and $d \in \mathbb{Z}^*$)

4. Assume $31 = ac_1 + bc_2$ and $\gcd(a,b) \neq 1$. Find $\gcd(a,b)$ [of course, a, b in $\mathbb{N}^*$ and $c_1, c_2 \in \mathbb{Z}$]

5. Convince me that $1963.2018\overline{2017}$ is a rational number

6. [Optional] Choose a number $n \in \mathbb{N}^*$. Convince me that there is a positive integer k, such that $k, k+1, k+2, k+3, \dots k+n$ are non-prime integers.

Answers:

1. n, m are distinct prime integers. This means $\gcd(n,m) = 1$

$1 = na_1 + ma_2$, $a_1, a_2 \in \mathbb{Z}$

multiply by 2018, hence ~~1 = n~~ $2018 = n(2018a_1) + m(2018a_2)$

let $2018a_1 = c_1$ and $2018a_2 = c_2$

Hence, $2018 = nc_1 + mc_2$, $c_1, c_2 \in \mathbb{Z}$ (integer x integer)

2. Assuming $m|a$ and $m|b$, ~~m~~ let $\cancel{\frac{m}{a} = n}$ $\frac{a}{m} = n$ and $\frac{b}{m} = k$, $n, k \in \mathbb{Z}$

then $\frac{ai_1}{m} = \frac{a}{m} i_1 = ni_1$ and $\frac{bi_2}{m} = \frac{b}{m} i_2 = ki_2$, $ni_1, ki_2 \in \mathbb{Z}$.

then $\frac{ai_1 + bi_2}{m} = \frac{ai_1}{m} + \frac{bi_2}{m}$ (same denominator)

$= ni_1 + ki_2 \in \mathbb{Z}$ (integer + integer)

hence, $m | (ai_1 + bi_2)$ for $i_1, i_2 \in \mathbb{Z}$

3. $x, y \in \mathbb{Q}$. Then $x = \frac{x_1}{x_2}$, $y = \frac{y_1}{y_2}$, ~~$x_1, x_2, y_1, y_2 \in \mathbb{Z}$~~ $x_1, y_1 \in \mathbb{Z}$, $x_2, y_2 \in \mathbb{Z}^*$

$x+y = \frac{x_1}{x_2} + \frac{y_1}{y_2}$

$= \frac{y_2 x_1 + x_2 y_1}{y_2 x_2}$, $y_2 x_1 + x_2 y_1 \in \mathbb{Z}$ (integer multiplication & addition), $y_2 x_2 \in \mathbb{Z}^*$

let $y_2 x_1 + x_2 y_1 = c$

$y_2 x_2 = d$

then $x+y = \frac{c}{d}$, $c \in \mathbb{Z}$ & $d \in \mathbb{Z}^*$

4. We know that $ac_1 + bc_2 = m \gcd(a,b)$

$m\gcd(a,b) = 31$, 31 is a prime number, its only factors are 31 and 1.

this means $\gcd(a,b)$ can only be either 31 or 1. since it is given that $\gcd(a,b) \neq 1$, then $\gcd(a,b) = 31$.

BRUNNEN

5. $1963.2018\overline{2017} = x$

$10000x = 19632018.\overline{2017}$

$$\begin{array}{rl} 10000x - x = & 19632018.2017\ 2017\ 2017 \ldots \\ - & 1963.2018\ 2017\ 2017 \ldots \\ \hline & 19630054.9999 \end{array}$$

$9999x = 19630054.9999$

$x = \dfrac{196300549999}{99990000}$

Rewrite 2: if $m|a$ and $m|b$ then $\exists$ $n_1 \in \mathbb{Z}$ where $n_1 = \frac{a}{m}$ and $n_2 \in \mathbb{Z}$, $n_2 = \frac{b}{m}$

let $\gcd(n_1, n_2) = k$, $k \in \mathbb{Z}$.
then, $k = n_1 i_1 + n_2 i_2$

multiplying by m: $mk = \cancel{ai_1 + bi_2}\ mn_1 i_1 + mn_2 i_2$
$= m\frac{a}{m} i_1 + m\frac{b}{m} i_2$

$mk = ai_1 + bi_2$

Hence $m \mid (ai_1 + bi_2)$

6. let $n = 5$.
$10! + 5$, $10! + 6$, $10! + 7$, $10! + 8$, $10! + 9$, $10! + 10$

$\underbrace{5\left(\frac{10!}{5} + 1\right), 6\left(\frac{10!}{6} + 1\right) + 7\left(\frac{10!}{7} + 1\right) \ldots\ldots 10\left(\frac{10!}{10} + 1\right)}_{\text{definitely not prime.}}$ can't be sure about $10! + 11$ prime or not.

let $n = 12$
$24! + 12$, $24! + 13$, $24! + 14$, ... $24! + 24$

$\Rightarrow$ $12\left(\frac{24!}{12} + 1\right)$, $\underbrace{13\left(\frac{24!}{13} + 1\right), \ldots}$, $24\left(\frac{24!}{24} + 1\right)$

definitely not prime, bc. they can be factored.

generally can be written as:
$n \in \mathbb{Z}$,
hence $(2n)! + n, (2n)! + (n+1), \ldots (2n)! + 2n$. are not prime

then $k = (2n)! + n$
$k+1 = (2n)! + n + 1$
$\vdots$
$k+n = (2n)! + 2n = (2n)! + n + n$

11/02/18 Result: There are infinitely many primes.
Proof (by contradiction): We ~~proof~~ assume there are finitely many prime numbers, say $q_1, q_2, \ldots, q_m$ prime numbers

~~Rea~~ Recall: any number can be factored by prime numbers / is a product of prime no.
Let $n = q_1 q_2 \cdots q_m + 1$. Claim n cannot be factored as product of prime numbers
$\boxed{1}$ n cannot be ~~be~~ prime. Why? It is clear $n > q_i$ for every $1 \leq i \leq m$
$\boxed{2}$ $n = d_1 d_2 \cdots d_k$, for some $k \in \mathbb{N}^*$, where each d_i is a prime number.,
$(d_1, d_2, \ldots, d_k$ need not be distinct)

$n = d_1 d_2 \cdots d_k = q_1 q_2 \cdots q_m + 1$

note that each d_i is one of the ~~q_i's~~ q_i's.
but none of the q_i's is a factor of $q_1 q_2 \cdots q_m + 1$

i.e $\frac{d_1 d_2 \cdots d_k}{q_i}$ for some $i \in \mathbb{N}^*$
can be factored

but

$\frac{q_1 q_2 \cdots q_m + 1}{q_i}$ cannot be factored,

eg let $n = d_1 d_2 \cdots d_k = q_1 q_1 q_2 q_m = q_1 q_2 q_3 \cdots q_m + 1$
notice ; q_1 can be factored from $q_1 q_1 q_2 q_m$, but cannot be factored from $q_1 q_2 \cdots q_m + 1$

$\therefore$ n cannot be written as a product of the finite set of prime numbers assumed
Hence there are infinitely many prime numbers.

numbered eg: take finite set 2, 3, 5.
$q_1 q_2 q_3 + 1 = n = \cancel{30}\ 31$
write ~~30~~ 31 as product of primes $\Rightarrow$ cannot write it with 2, 3 or 5 factored.

~~Eg~~ Question . Convince me that $3x \equiv 4 \pmod 7$ iff $7 \mid (3x-4)$
Proof: Assume $3x \equiv 4 \pmod 7$. We show $7 \mid (3x-4)$ → direct.

$\Rightarrow$

$3x = 7q + 4$, 4 is remainder, q is quotient.
Rearranging ;
$3x - 4 = 7q$ Hence 7 is a factor.

$\Leftarrow$

Assume $7 \mid (3x-4)$. Show that $3x \equiv 4 \pmod 7$
This means $3x - 4 = 7m$ for some integer m.
Rearrange: $3x = 7m + 4$ (m is now quotient, ~~m~~ 4 is remainder)

$\Downarrow$

$3x \equiv 4 \pmod 7$

Know: $ax \equiv b \pmod m$ iff $m \mid (ax - b)$

Question: Convince me that $\sqrt{3}$ is not rational.
Proof: (by contradiction). Deny. Hence assume $\sqrt{3}$ is rational. Thus $\sqrt{3}$ can be written as $\sqrt{3} = \frac{a}{b}$, $\gcd(a,b) = 1$ (reduced form) and $a, b \in \mathbb{Z}$ and $b \neq 0$.

- Since $\gcd(a,b) = 1$, we can deduce that both a & b cannot be even (bc. gcd of even pair is ≥ 2).

Square both sides : $3 = \frac{a^2}{b^2}$

$\Rightarrow 3b^2 = a^2$, if a was even and b odd, $3b^2$ would be odd and a^2 would be even. even $\neq$ odd, contradiction. Similarly, if a was odd and b even, $3b^2$ would be even and a^2 ~~would~~ would be odd. Again ~~a~~ odd $\neq$ even, contradiction.
Only possibility : both a and b are odd.

So, let $a = 2m+1$, $m \in \mathbb{Z}$
and let $b = 2k+1$, $k \in \mathbb{Z}$

$a^2 = 4m^2 + 4m + 1$, $b^2 = 4k^2 + 4k + 1$

From $3b^2 = a^2$, $12k^2 + 12k + 3 = 4m^2 + 4m + 1$
Remove 1 from each, $12k^2 + 12k + 2 = 4m^2 + 4m$
Divide by 4, $\underbrace{3k^2 + 3k + \frac{1}{2}}_{\text{not an integer}} = \underbrace{m^2 + m}_{\text{purely integer.}}$

Hence, contradiction: $3k^2 + 3k + \frac{1}{2} \neq m^2 + m$.
Therefore, $3 \neq \frac{a^2}{b^2} \rightarrow \sqrt{3} \neq \frac{a}{b}$.

$\therefore \sqrt{3}$ must be irrational.

Technique: choose any prime q, where 4 is not a factor of $(q-1)$. This will work to show that $\sqrt{q}$ is irrational.
eg: $q = 11$

$$11b^2 = a^2$$
$$11(4k^2 + 4k + 1) = 4m^2 + 4m + 1$$
$$44k^2 + 44k + 10 = 4m^2 + 4m$$
$$\underbrace{11k^2 + 11k + \frac{10}{4}}_{\text{not integer}} = \underbrace{m^2 + m.}_{\text{integer.}}$$

Question. Prove that $\sqrt{3}$ is irrational (Method 2).

Facts from Number theory:
q is prime
$n \in \mathbb{Z}$
If $q^k \mid n^w$ for some positive integers k, w, then $q \mid n$

assume $7^3 \mid n^{100}$, can conclude $7 \mid n$.
or $11^{20} \mid n$, can conclude $11 \mid n$.

* q has to be prime for this to work. If q is not prime, $q^k \mid n^w \rightarrow q \mid n$ is not true.

Additionally, $q \mid n$ means $n = mq$, $m \in \mathbb{Z}$. (n is a multiple of q).

Back to question: Show that $\sqrt{q}$ is irrational, for any prime q.
Proof: (by contradiction). Deny. Assume $\sqrt{q}$ is rational.
Thus $\sqrt{q} = \frac{a}{b}$, for $a \in \mathbb{Z}$, $b \in \mathbb{Z}^*$

Square both sides (to change everything to planet $\mathbb{Z}$)
$q = \frac{a^2}{b^2} \Rightarrow * \; qb^2 = a^2$
$\Rightarrow q \mid a^2$
$\Rightarrow q \mid a$ (Number Theory above)

Hence $a = qw$, $w \in \mathbb{Z}$ (i.e. a is a multiple of q).
$** \; a^2 = q^2w^2$

Substitute back: From $(*)$ and $(**)$, $qb^2 = q^2w^2$
Divide by q: $b^2 = qw^2$
conclude: $q \mid b$
But, $\frac{a}{b}$ implies that $\gcd(a,b) = 1$. If $q \mid a$ and $q \mid b$, then $\gcd(a,b) \neq 1$ (Contradiction).

Hence, $\sqrt{q}$ cannot be rational.
i.e. $\sqrt{q}$ is irrational.
i.e $\sqrt{3}$ is also irrational.

Question: Show $\sqrt{3}+1$ is Irrational.
Proof (by contradiction): Deny. Assume $\sqrt{3}+1$ is rational.

$\sqrt{3}+1 = \frac{a}{b}$, $a, b \in \mathbb{Z}$, ~~be~~ $b \neq 0$, $\gcd(a,b) = 1$.

$\sqrt{3} = \frac{a}{b} - 1$. // isolate the squareroot.

From the exercise above, $\sqrt{3}$ is irrational. And $\frac{a}{b} - 1$ is rational.

Rational $\neq$ Irrational.
Hence ~~$\sqrt{q}+$~~ $\sqrt{3}+1$ is Irrational.
Generally, $\sqrt{q} + r$, q is prime and $r \in \mathbb{Q}$, the result is irrational.

$\boxed{1}$ Let a be an odd number. Show a^2 is odd number.

$\boxed{2}$ Let a be an even integer. Show a^2 is even integer.

$\boxed{3}$ Show $\sqrt{12}$ is irrational. (Hint: isolate the prime).

$\boxed{4}$ Show that $\sqrt{45}$ is irrational. (Hint: see question above).

$\boxed{5}$ Show $\sqrt{15}$ is irrational. (Try to use method 1).

$\boxed{6}$ Irrational + Irrational = irrational. True or false. If false, give counter example.

Solutions: $\boxed{1}$ Let $a = 2m+1$, $m \in \mathbb{Z}$

$$a^2 = (2m+1)^2$$
$$= 4m^2 + 4m + 1$$
$$= 2(2m^2 + 2m) + 1$$

let $2m^2 + 2m = k$, $k \in \mathbb{Z}$

Hence, $a^2 = 2k + 1 \Rightarrow$ odd.

$\therefore$ square of an odd number is always odd.

$\boxed{2}$ let $a = 2n$, $n \in \mathbb{Z}$.

$a^2 = (2n)^2 = 4n^2 = 2(2n^2)$.

let $2n^2 = w$, $w \in \mathbb{Z}$

Hence, $a^2 = 2w \Rightarrow$ even.

$\therefore$ square of an even number is always even.

$\boxed{3}$ ~~t_2~~ $\sqrt{12} = \sqrt{2 \times 2 \times 3} = 2\sqrt{3}$

(~~3 is a prime number, hence $\sqrt{3}$ is irrational.~~)

Deny. Say $2\sqrt{3}$ is a rational number.

Hence, $2\sqrt{3} = \frac{a}{b}$, $a \in \mathbb{Z}$, ~~&~~ $b \in \mathbb{Z}^*$

$$\sqrt{3} = \frac{a}{2b}$$

~~Squar~~ but $\sqrt{3}$ has been established as an irrational number,

irrational $\neq$ ~~$\neq$~~ rational (contradiction).

Hence $2\sqrt{3}$ i.e $\sqrt{12}$ is an irrational number.

4 $\sqrt{45} = \sqrt{5 \times 3 \times 3} = 3\sqrt{5}$

Deny. Say $3\sqrt{5}$ is rational.

$$3\sqrt{5} = \frac{a}{b}, \quad a,b \in \mathbb{Z}, \ b \neq 0$$

$$\sqrt{5} = \frac{a}{3b}.$$

but $\sqrt{5}$ is an irrational number.

irrational $\neq$ rational. (contradiction)

Hence $3\sqrt{5}$ i.e $\sqrt{45}$ is an irrational number.

5 Deny. Say $\sqrt{15}$ is ~~irrational~~ rational.

$$\sqrt{15} = \frac{a}{b}, \quad a,b \in \mathbb{Z}, \ b \neq 0$$

Square both sides: $15 = \dfrac{a^2}{b^2}$

We have established that a and b are odd numbers.

So let $a = 2m+1$, $m \in \mathbb{Z}$

$b = 2k+1$, $k \in \mathbb{Z}$

$$15(4k^2+4k+1) = 4m^2+4m+1$$

$$60k^2+60k+15 = 4m^2+4m+1$$

$$\Rightarrow 60k^2+60k+14 = 4m^2+4m.$$

Divide by 4 $\Rightarrow$ $\underbrace{15k^2+15k+\frac{14}{4}}_{\text{not integer as } 4 \nmid 14} = \underbrace{m^2+m}_{\text{integer}}$

Hence $15k^2+15k+\frac{7}{2} \neq m^2+m$

So we ~~can~~ conclude $\sqrt{15}$ is not rational, i.e irrational.

6 Irrational + irrational = ~~irrat~~ irrational is false.

Counter example; $(\sqrt{2}+1) + (-\sqrt{2}) = 1$, $1 \in \mathbb{Q}$ & $1 \in \mathbb{Z}$

15/02/18. General result: (not allowed to use in exam)

let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$ (product of primes)

if any of the α_i's is odd (at least 1), then $\sqrt{n}$ is irrational.

eg: $147 = 3^1 \times 7^2$

Hence $\sqrt{147}$ is irrational.

From this, we conclude product of $\sqrt{q}$ and $\sqrt{r}$ where q, r are distinct primes is irrational.

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